Solution to the Final Exam of M2CNES

2021 - 2022

Problem 1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_2^2 + x_3^2 \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u.$$

Define $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $f(x) = \begin{bmatrix} x_2 + x_2^2 + x_3^2 \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 \end{bmatrix}$, and

$g(x) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, we have

$$\dot{x} = f(x) + g(x) u.$$

a) Now we need to define three independent vectors that belong to the strong accessibility algebra $\mathcal{C}_{sa}$. To this end, we compute

$$[f, g] = \frac{\partial g}{\partial x}(x) f(x) - \frac{\partial f}{\partial x}(x) g(x)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 + x_2^2 + x_3^2 \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 1 + 2x_2 & 2x_3 \\ \cos(x_1 - x_3) & 0 & 1 - \cos(x_1 - x_3) \\ 0 & 0 & 2x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2x_3 \\ -1 \\ -2x_3 \end{bmatrix}$$

$$[f,[f,g]] = \frac{\partial [f,g]}{\partial x}(x) f(x) - \frac{\partial f}{\partial x}(x)\,[f,g](x)$$

$$= \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_2 + x_2^2 + x_3^2 \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 1+2x_2 & 2x_3 \\ \cos(x_1 - x_3) & 0 & 1-\cos(x_1-x_3) \\ 0 & 0 & 2x_3 \end{bmatrix} \begin{bmatrix} -2x_3 \\ -1 \\ -2x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2x_2+2x_3^2 \\ 2x_3 \\ 2x_3^2 \end{bmatrix}$$

Note that g, $[f,g]$, $[f,[f,g]]$ are contained in the strong accessibility algebra $\mathcal{C}_{sa}$.

Let us construct a matrix $M_{\mathcal{C}_{sa}}$ with these vector fields as

$$M_{\mathcal{C}_{sa}} = \begin{bmatrix} 1 & -2x_3 & 1+2x_2+2x_3^2 \\ 0 & -1 & 2x_3 \\ 1 & -2x_3 & 2x_3^2 \end{bmatrix}$$

whose determinant is $1+2x_2$. Therefore, $\mathcal{C}_{sa}$ contains $n=3$ independent vector fields if $x_2 \neq -1/2$. It follows that the system is locally strongly accessible for $x_2 \neq -1/2$.

b) We first check if the vector fields g, $ad_f g$ and $ad_f^2 g$ are independent near x_0. To this end, we have from part (a):

$$ad_f g = [f, g] = \begin{bmatrix} -2x_3 \\ -1 \\ -2x_3 \end{bmatrix}$$

and $ad_f^2 g = [f, ad_f g] = [f, [f, g]] = \begin{bmatrix} 1+2x_2+2x_3^2 \\ 2x_3 \\ 2x_3^2 \end{bmatrix}$

We define

$$M(x) = [g(x)\ ad_f g(x)\ ad_f^2 g(x)]$$

$$= \begin{bmatrix} 1 & -2x_3 & 1+2x_2+2x_3^2 \\ 0 & -1 & 2x_3 \\ 1 & -2x_3 & 2x_3^2 \end{bmatrix}$$

$$\det M(x) = 1 + 2x_2$$

which implies that the vector fields are independent for $x_2 \neq -1/2$.

Now we need to check if

$$\Delta(x_0) = \text{span}\{g, ad_f g\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2x_{30} \\ -1 \\ -2x_{30} \end{bmatrix} \right\}$$

is involutive. To this end, we compute

$$[g, ad_f g] = -\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 0 = -\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = -2g \in \Delta.$$

Hence, $\Delta(x_0)$ is involutive, therefore, the system is linearizable to a controllable linear system.

c) Consider the output $y = x_2$. We have

$$L_g h(x) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$L_f h(x) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 + x_2^2 + x_3^2 \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 \end{bmatrix}$$

$$= x_3 + \sin(x_1 - x_3)$$

So,

$$L_g L_f h(x) = \begin{bmatrix} \cos(x_1 - x_3) & 0 & 1 - \cos(x_1 - x_3) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1$$

So, relative degree for this system ρ is 2.

d) Now the input-output linearization is achieved for

$$u = \frac{1}{L_g L_f h(x)} \left(-L_f^2 h(x) + v\right)$$

$$u = -L_f^2 h(x) + v \quad \text{as } L_g L_f h(x) = 1.$$

Now

$$L_f^2 h = \begin{bmatrix} \cos(x_1 - x_3) & 0 & 1 - \cos(x_1 - x_3) \end{bmatrix} \begin{bmatrix} x_2 + x_2^2 + x_3^2 \\ x_3 + \sin(x_1 - x_3) \\ x_3^2 \end{bmatrix}$$

$$= \cos(x_1 - x_3)(x_2 + x_2^2 + x_3^2) + x_3^2 - x_3^2 \cos(x_1 - x_3)$$

Hence,

$$u = -(x_2 + x_2^2)\cos(x_1 - x_3) - x_3^2 + v$$

Now for reference trajectory tracking, we have

$$v = y_{ref}^{(2)}(t) - \alpha_1\left(y^{(1)}(t) - y_{ref}^{(1)}(t)\right) - \alpha_0\left(y(t) - y_{ref}(t)\right)$$

$$= -\omega^2\cos(\omega t) - \alpha_1\left(x_3 + \sin(x_1 - x_3) + \omega\sin(\omega t)\right)$$

$$- \alpha_0\left(x_2 - \cos(\omega t)\right)$$

Choosing, for instance, $\alpha_0 = 1$ and $\alpha_1 = 2$ gives a control law

$$u = -(x_2 + x_2^2)\cos(x_1 - x_3) - x_3^2 - \omega^2\cos(\omega t)$$

$$-2\left(x_3 + \sin(x_1 - x_3) + \omega\sin(\omega t)\right)$$

$$-\left(x_2 - \cos(\omega t)\right)$$

that tracks the reference trajectory $y_{ref}(t) = \cos(\omega t)$ for some $\omega > 0$.

Problem 2

(a) Consider

$$\dot{x} = (1+\epsilon)x^2 + u$$

with $u = -x^2 - x + v$ for $v = 0$. Then the perturbed closed-loop system is given by

$$\dot{x} = (1+\epsilon)x^2 - x^2 - x$$

$$= \epsilon x^2 - x$$

Note that for $x > 1/\epsilon$, we have $\dot{x} > 0$, which implies that the trajectories tend to infinity. Thus, global cancellation is non-robust in the sense that it requires a very precise mathematical model.

b) Now consider $\dot{x} = (1+\epsilon)x^2 + u$ with $u = -x^2 - x - kx^3 + v$ with $v = 0$, then the perturbed closed-loop system is given by

$$\dot{x} = \epsilon x^2 - x - kx^3.$$

Now for any non-zero ϵ, if we choose k such that $kx^3 > \epsilon x^2 - x$, then the trajectories tend to zero as $\dot{x} < 0$.

Problem No 3

$$H = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} q^T \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix} q,$$

where

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}}_{J = -J^T} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u.$$

Let $x = \begin{bmatrix} q \\ p \end{bmatrix}$ and take H as storage function, then

$$\dot{H} = \frac{\partial H}{\partial x}^T \dot{x} = \underbrace{\frac{\partial H}{\partial x}^T J \frac{\partial H}{\partial x}}_{\substack{=0 \text{ as } J \text{ is} \\ \text{skew-symmetric}}} + \frac{\partial H}{\partial p}^T u = \frac{\partial H}{\partial p}^T u = y^T u,$$

where

$$y = \frac{\partial H}{\partial p} = \dot{q} = M^{-1} p.$$

b) Now let $u = -\underbrace{\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}}_{D} M^{-1}p + v$ with $d_1 \geq 0$ and $d_2 \geq 0$. It follows that

$$u = -DM^{-1}p + v = -D\dot{q} + v = -D\frac{\partial H}{\partial p} + v.$$

Now

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ -D\frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v$$

$$= \begin{bmatrix} 0 & I \\ -I & -D \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v$$

Let $R = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \geq 0$. Then

$$\dot{x} = (J-R)\frac{\partial H}{\partial x} + Gv,$$

where $G = \begin{bmatrix} 0 \\ I \end{bmatrix}$. It follows that

$$\dot{H} = \frac{\partial H}{\partial x}^T \dot{x} = -\frac{\partial H}{\partial x}^T R \frac{\partial H}{\partial x} + \frac{\partial H}{\partial p}^T v \leq \frac{\partial H}{\partial p}^T v = y^T v,$$

with $y = \frac{\partial H}{\partial p} = \dot{q} = M^{-1}p.$

Yes, H is still a storage function.

Problem No 4 (For IEM and ME students)

The sign conditions given in this problem are:

$$QC(Q) > 0 \quad \forall Q \neq 0$$

$$IR(I) > 0 \quad \forall I \neq 0$$

(a) Together with continuity and the above sign conditions on $C(Q)$ and $R(I)$ imply that

$$C(0) = 0 \text{ and } R(I) = 0.$$

Now in order to find the equilibrium points, we have that

$$\dot{Q} = 0 = I \text{ and } \dot{I} = 0 = -L^{-1}C(Q) - L^{-1}R(0) = 0.$$

This implies that $C(Q) = 0$, which further implies that $Q = 0$. Hence, $x = 0$ is the unique equilibrium point.

b) Consider $V(x) = \frac{1}{2}LI^2 + \int_0^Q C(q)\,dq$. Note that $V(0) = 0$. Also note that $\frac{1}{2}LI^2 > 0$ for $I \neq 0$ because $L > 0$. Note that $QC(Q) > 0$ for $Q \neq 0$ implies that $C(Q) > 0$ for $Q > 0$ and $C(Q) < 0$ for $Q < 0$. Hence, if $Q > 0$ then $\int_0^Q C(q)\,dq > 0$ and if $Q < 0$ then $\int_0^Q C(q)\,dq > 0$. This implies that $V(x) > 0$ for $x \neq 0$.

c) $\dot{V} = LI\dot{I} + C(Q)\dot{Q}$

$= LI\left(-L^{-1}C(Q) - L^{-1}R(I)\right] + C(Q)I$

$= -C(Q)I - IR(I) + C(Q)I$

$= -IR(I) \leq 0.$

Hence, origin is stable (in sense of Lyapunov).

d) Note that $IR(I) = 0 \Rightarrow I = 0$. Now

$I = 0 \Rightarrow \dot{I} = 0 \Rightarrow -L^{-1}C(Q) - L^{-1}R(0) = 0$

$\Rightarrow -L^{-1}C(Q) = 0 \Rightarrow Q = 0.$

Hence, by LaSalle's invariance principle, the origin is asymptotically stable.

e) If the integral $\int_0^Q C(q)\,dq$ is unbounded as $|Q| \to \infty$, then V is radially unbounded.

Scanned with CamScanner

Problem 4 (Mathematics Students)

a) Assume that there exists a diagonal positive definite matrix P such that

$$(\Gamma K)^T P (\Gamma K) - P < 0.$$

Write

$$P = \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_N \end{bmatrix}$$

and consider the function

$$V(x) = \sum_{i=1}^{N} P_i S_i(x_i).$$

Note that since each $S_i \geq 0$, We have $V(x) \geq 0$.

Then, we have that

$$\dot{V}(x) = \sum_{i=1}^{N} P_i \dot{S}_i(x_i)$$

$$\leq \sum_{i=1}^{N} P_i \left(\frac{1}{2} \gamma_i^2 \|u_i\|^2 - \frac{1}{2} \|y_i\|^2 \right)$$

$$= \frac{1}{2} \sum_{i=1}^{N} P_i \gamma_i^2 u_i^2 - \frac{1}{2} \sum_{i=1}^{N} P_i y_i^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} u_i \gamma_i P_i \gamma_i u_i - \frac{1}{2} \sum_{i=1}^{N} P_i y_i^2$$

$$= \frac{1}{2} (\Gamma u)^T P (\Gamma u) - \frac{1}{2} y^T P y$$

Using $u=Ky$, we have

$$\dot{V} \leq \frac{1}{2} y^T (\Gamma K)^T P (\Gamma K) y - \frac{1}{2} y^T P y$$

$$= \frac{1}{2} y^T ((\Gamma K)^T P (\Gamma K) - P) y$$

$$< 0.$$

Therefore, based on zero state observability, the origin is asymptotically stable.

b) From part (a), we have

$$\dot{V} \leq \frac{1}{2} (\Gamma u)^T P (\Gamma u) - \frac{1}{2} y^T P y.$$

Now with the choice of $u = Ky + Lv$ and $z = My$, we want to have

$$\dot{V} \leq \frac{1}{2} (\Gamma K y + \Gamma L v)^T P (\Gamma K y + \Gamma L v) - \frac{1}{2} y^T P y \leq \frac{1}{2} \theta^2 v^T v - \frac{1}{2} y^T M^T M y.$$

It follows that

$$\frac{1}{2} (\Gamma K y)^T P (\Gamma K y) + \frac{1}{2} (\Gamma K y)^T P (\Gamma L v) + \frac{1}{2} (\Gamma L v)^T P (\Gamma L v) + \frac{1}{2} (\Gamma L v)^T P (\Gamma K y)$$
$$- \frac{1}{2} y^T P y - \frac{1}{2} \theta^2 v^T v + \frac{1}{2} y^T M^T M y \leq 0.$$

or

$$\frac{1}{2} [y^T \ v^T] \begin{bmatrix} K^T \Gamma^T P \Gamma K - P + M^T M & K^T \Gamma^T P \Gamma L \\ L^T \Gamma^T P \Gamma K & L^T \Gamma^T P \Gamma L - \theta^2 I \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} \leq 0.$$

So a sufficient condition in terms of matrix inequality is:

$$\begin{bmatrix} K^T \Gamma^T P \Gamma K - P + M^T M & K^T \Gamma^T P \Gamma L \\ L^T \Gamma^T P \Gamma K & L^T \Gamma^T P \Gamma L - \theta^2 I \end{bmatrix} \prec 0.$$